# Propagation of combined standard uncertainties evaluated in SPRT calibration according to ITS-90

Sonia GAIŢĂ
*Romanian Bureau of Legal Metrology - National Institute of Metrology*
*Şos. Vitan-Bârzeşti 11, 75669 Bucureşti, ROMÂNIA*
*Phone: 40 94 35 22 63, Fax: 40 1 334 53 45, E-mail: gaita@inm.ro*

Constantin ILIESCU
*University "Politehnica" of Bucharest – Faculty of Electrical Engineering*
*Splaiul Independenţei 313, EB Building, Bucureşti, ROMÂNIA*

**Abstract.** *The paper presents a model for evaluating the combined standard uncertainty at any intermediate temperature, propagated from the SPRT calibration uncertainties at the defining fixed points of the ITS-90. The new proposed expressions of $W_{FP}$ and $W(T_{90})$ are based on a thorough and systematic approach in dealing with the correlations issue. They also have the advantage of enabling the outline of the contributory variances of the genuine input quantities at any temperature within the calibration range of the SPRT. Computer processing allowed all variables and their correlations to be included and treated in a consistent manner. The application of the model is illustrated for the temperature sub-range from 273.15 K to 692.677 K.*

## 1. Introduction

The Standard Platinum Resistance Thermometer (SPRT) is the interpolating instrument specified in the International Temperature Scale of 1990 (ITS-90) [1] for the definition of the $T_{90}$ temperatures in the eleven sub-ranges from 13.803 3 K to 961.78 ºC.

SPRT is calibrated at those definition fixed points specified in ITS-90 for the sub-range in which it is used. The values of the SPRT resistance obtained from measurements are used to determine the values of the ratios:

$$W(T_{90}) = R(T_{90}) / R(273.16K) \tag{1}$$

at any intermediate temperature.

The algorithm given by ITS-90 for determining the SPRT characteristic makes that the calibration uncertainties propagate in the uncertainties of the $W(T_{90})$ ratios determined at the fixed points and further, in the uncertainties of those ratios determined at any intermediate temperature. But the values of the propagated uncertainties depend in an essential manner of both the analytical expressions of the inputs for the algorithm, and the correlations among those inputs.

## 2. The mathematical model

### 2.1 The model functions

To determine the characteristic $W=f(T_{90})$ of an SPRT, one uses the specified reference functions $W_r(T_{90})$ and the deviation functions $\Delta W(T_{90})$:

$$W(T_{90}) = W_r(T_{90}) + \Delta W(T_{90}). \tag{2}$$

ITS-90 defines two reference functions $W_r(T_{90})$: a logarithmic one – for temperatures under 0.01 ºC, and a polynomial one – for temperatures over 0 ºC, their coefficients being listed in Table 4 of [1]. The deviation function $\Delta W$ is defined separately for each sub-range of ITS-90. It is a function with a general form, but the values of its coefficients are specific to the SPRT under calibration and are determined by SPRT calibration.

Let $R_{TPW}$ be the SPRT resistance measured at the triple point of water (TPW) and $R_{FP}$ be the SPRT resistance measured at any other fixed point (FP) in the calibration range of the SPRT. The measurement at TPW has to be repeated after the measurement at each FP, so that the ratio $W_{FP} = \frac{R_{FP}}{R_{TPW}}$ be determined by a pair of values $R_{FP}$ and $R_{TPW}$ corresponding to the same physical and chemical status of the sensor [2]. This is one reason for which the calibration uncertainty at the fixed points is better described by $u_c(W_{FP})$ - the combined standard uncertainty (CSU) associated with $W_{FP}$, than by $u_c(R_{FP})$. It is $u_c(W_{FP})$ that is used for the international comparison of the national realizations of the ITS-90 in the framework of MRA (Mutual Recognition Arrangement), aiming at evaluating the equivalence among national standards.

But the inputs $R_{FP}$ and $R_{TPW}$ are correlated, as they depend upon several common variables. This correlation is difficult to assess and therefore it is preferably to expand out the expressions of $R_{FP}$ and $R_{TPW}$ in terms of the input quantities they depend upon [3].

An essential feature of the expressions of $R_{TPW}$ and $R_{FP}$ developed in [3] and took over here is that the input quantities are independent, thus eliminating the necessity of evaluating the correlations among them. There are, still, two input quantities correlated on technical grounds, $I_1$ and $I_2$, but their correlation coefficient can be approximated reasonably well by 1.

$W_{FP}$ turns this way into a function of 34 input quantities $X_i$ (16 in the expression of $R_{TPW}$ and 18 in expression of $R_{FP}$):

$$
\begin{aligned}
W_{FP} = {} & (C_{1/FP} + C_{2/FP} - A_{FP}\, h_{FP} - B_{FP}\, \delta p_{FP}) / K \left.\frac{dW_r}{dT_{90}}\right|_{T_{90}=T_{FP}} + \\
& \frac{\{[1+\alpha_{FP}(T_{b1/FP} - T_r)]\, r_{1/FP}\, r_{c1/FP}\, I^2_{2/FP} - [1+\alpha_{FP}(T_{b2/FP} - T_r)]\, r_{2/FP}\, r_{c2/FP}\, I^2_{1/FP}]\}}{\{[1+\alpha_{TPW}(T_{b1/TPW} - T_r)]\, r_{1/TPW}\, r_{c1/TPW} I^2_{2/TPW} - [1+a_{TPW}(T_{b2/TPW} - T_r)]\, r_{2/TPW}\, r_{c2/TPW} I^2_{1/TPW}\}} \times \\
\times\, & \frac{R_{s/FP}\,[1+b_{FP}(t_{FP}-t_0)]\left[1-(C_{1/TPW}+C_{2/TPW}-A_{TPW}\, h_{TPW})/K \left.\frac{dW_r}{dT_{90}}\right|_{T_{90}=273.16\ K}\right](I^2_{2/TPW} - I^2_{1/TPW})}{R_{s/TPW}\,[1+b_{TPW}(t_{TPW}-t_0)]\,(I^2_{2/FP} - I^2_{1/FP})}
\end{aligned}
\qquad (3)
$$

where the notations for input quantities are:

$R_{s/TPW}$, $R_{s/FP}$ – the resistance of the standard resistor;

$b_{TPW}$, $b_{FP}$ – the coefficient of the drift of the resistance of the standard resistor since its latest calibration;

$t_{TPW}$, $t_{FP}$ – the time of the calibration of SPRT at TPW and FP, respectively;

$t_0$ – the time of the calibration of the standard resistor ( $\{t_0\}_{\mathrm{d}} = 0$ );

$\alpha_{TPW}$, $\alpha_{FP}$ – the temperature coefficient of the standard resistor;

$T_{b1/TPW}$, $T_{b1/FP}$, $T_{b2/TPW}$, $T_{b2/FP}$ – the temperatures of the oil bath for the maintenance of the standard resistor during measurements, using the currents $I_1$ and $I_2$, respectively;

$T_r$ – the calibration temperature of the standard resistor ( $\{T_r\}_{\mathrm{K}} = 293.15$ );

$r_{1/TPW}$, $r_{1/FP}$; $r_{2/TPW}$, $r_{2/FP}$ – the readings of the bridge for the currents $I_1$ and $I_2$, respectively;

$r_{c1/TPW}$, $r_{c1/FP}$; $r_{c2/TPW}$, $r_{c2/FP}$ – the correction factors for the readings $r_1$ and $r_2$, respectively;

$I_{1/TPW}$, $I_{1/FP}$, $I_{2/TPW}$, $I_{2/FP}$ – the measurement currents of the bridge;

$A_{FP}$, $A_{TPW}$ – the coefficient of variation of the temperature with the immersion depth;

$h_{FP}$, $h_{TPW}$ - the immersion depth;

$B_{FP}$ – the coefficient of variation of the FP temperature with the deviation of the gas pressure in the cell from the reference pressure [1];

$\delta p_{FP}$ – the deviation of the gas pressure in the cell from the reference pressure.

For simplicity of notations, the same symbol is used for a quantity and for its estimate.

The model functions for the evaluation of the CSU associated with $W(T_{90})$ are the interpolation equations (2). At a given temperature within the SPRT calibration range, $W_r(T_{90})$ is a constant and $\Delta W(T_{90})$ is [1] a function $f_1$ of the ratios $W_{FP}$ determined at the $n$ fixed points specified for that range:

$$\Delta W(T_{90}) = f_1(W_{FP1}, W_{FP2}, ..., W_{FPn}) \tag{4}$$

As the ratios $W_{FPk}$, k=1,2, …, $n$ are correlated, the aspects highlighted above are valid here as well, so $W(T_{90})$ will be expressed as a function $f_2$ of the input quantities $X_{i,k}$ of all ratios $W_{FPk}$ determined at the $n$ fixed points in the SPRT calibration range:

$$W(T_{90}) = f_2(X_{i,k}), \quad i = 1, 2, ..., 34; \quad k = 1, 2, ..., n. \tag{5}$$

### 2.2 The combined standard uncertainty

In order to evaluate the CSUs associated with $W_{FP}$ and, respectively, with $W(T_{90})$, one needs first to determine the correlations among the quantities used as inputs for this evaluation. A distinction is made in this model between "intrinsic" and "inherent" uncertainties associated with the estimates of the input quantities:

- quantities associated only with an inherent uncertainty materialize the same value, regardless the measurement that they refer to, at an unknown position within the uncertainty range;
- quantities associated with an intrinsic uncertainty materialize potentially different values at different measurements, within the uncertainty range.

Important consequences follow regarding both the correlations among these quantities and the strategic use of notations to avoid premature reduction in the symbolic manipulation phase of the CSU computing process. Details are provided in [7].

In case all these correlations are considered, the expression of $W_{FP}$ can be simplified as follows:

- $R_s$ and the entire drift of the standard resistor (for TPW and for FP) are reduced, and
- $\alpha_{TPW}$ and $\alpha_{FP}$ receive the same symbol ($\alpha$).

The CSU associated with $W_{FP}$ is determined by the law of propagation of uncertainty for correlated input quantities [4], that becomes:

$$\begin{aligned} u_c^2(W_{FP}) = \sum_{i=1}^{23}\left(\frac{\partial f_3}{\partial x_i}\right)^2 u^2(x_i) + \left(\frac{\partial f_3}{\partial I_{1/TPW}} u(I_{1/TPW}) + \frac{\partial f_3}{\partial I_{2/TPW}} u(I_{2/TPW})\right)^2 \\ + \left(\frac{\partial f_3}{\partial I_{1/FP}} u(I_{1/FP}) + \frac{\partial f_3}{\partial I_{2/FP}} u(I_{2/FP})\right)^2 \end{aligned} \tag{6}$$

where $f_3$ is $W_{FP}$ in (3) as function of the input estimates $x_1, x_2, ..., x_{23}, I_{1/TPW}, I_{2/TPW}, I_{1/FP}, I_{2/FP}$.

The number of variables in the expression of $W(T_{90})$ in each sub-range of ITS-90 depends on the number $n$ of fixed points in that sub-range. The CSU associated with $W(T_{90})$ is determined with the relation:

$$\begin{aligned} u_c^2(W) = \sum_{i=1}^{N}\left(\frac{\partial f_2}{\partial x_i}\right)^2 u^2(x_i) \\ + \sum_{k=1}^{n}\left[\left(\frac{\partial f_2}{\partial I_{1/TPWk}} u(I_{1/TPWk}) + \frac{\partial f_2}{\partial I_{2/TPWk}} u(I_{2/TPWk})\right)^2 + \left(\frac{\partial f_2}{\partial I_{1/FPk}} u(I_{1/FPk}) + \frac{\partial f_2}{\partial I_{2/FPk}} u(I_{2/FPk})\right)^2\right] \end{aligned} \tag{7}$$

where N=22n+1 is the number of independent input estimates $x_i$.

This way, in the present approach, unlike in all other reported models, with the exception of [5], $W_{FP}$ and $W(T_{90})$ are eventually expressed as functions of the genuine input quantities of $R_{TPW}$ and $R_{FP}$, that are either quantities directly measured, or quantities directly characterized using scientific judgement.

The resulting forms of $W_{FP}$ and $W(T_{90})$ are based on a systematic and consistent approach in dealing with the correlations issue. They also have the advantage of enabling the outline of the contributory variances of the genuine input quantities at any temperature within the calibration range of the SPRT.

**Table 1.** Input data

| Quantity | Estimate | Standard uncertainty |
|---|---|---|
| $R_s$ | 9.999 947 Ω | 3 x $10^{-6}$ Ω |
| $b$ | -5.48x$10^{-10}$ $d^{-1}$ | 1.92x$10^{-10}$ $d^{-1}$ |
| $\alpha$ | 16.36 x$10^{-6}$ $K^{-1}$ | 6 x$10^{-8}$ $K^{-1}$ |
| $T_{b1/Zn}$, $T_{b2/Zn}$ | 293.169 K | 0.007 K |
| $T_{b1/TPW1}$, $T_{b2/TPW1}$ | 293.171 K | 0.007 K |
| $T_{b1/Sn}$, $T_{b2/Sn}$ | 293.174 K | 0.007 K |
| $T_{b1/TPW2}$, $T_{b2/TPW2}$ | 293.175 K | 0.007 K |
| $r_{1/Zn}$ | 6.556 916 2 | 15 x $10^{-7}$ |
| $r_{2/Zn}$ | 6.556 947 4 | 15 x $10^{-7}$ |
| $r_{1/TPW1}$ | 2.552 565 4 | 3 x $10^{-7}$ |
| $r_{2/TPW1}$ | 2.552 595 9 | 3 x $10^{-7}$ |
| $r_{1/Sn}$ | 4.831 262 5 | 13 x $10^{-7}$ |
| $r_{2/Sn}$ | 4.831 294 4 | 13 x $10^{-7}$ |
| $r_{1/TPW2}$ | 2.552 565 6 | 3 x $10^{-7}$ |
| $r_{2/TPW2}$ | 2.552 596 2 | 3 x $10^{-7}$ |
| $r_{c1/Zn}$, $r_{c2/Zn}$, $r_{c1/TPW1}$, $r_{c2/TPW1}$, $r_{c1/Sn}$, $r_{c2/Sn}$, $r_{c1/TPW2}$, $r_{c2/TPW2}$ | 1.000 000 0 | 1 x $10^{-7}$ |
| $I_{1/Zn}$, $I_{1/TPW1}$, $I_{1/Sn}$, $I_{1/TPW2}$ | 1.000 x $10^{-3}$ A | 1.6 x $10^{-5}$ A |
| $I_{2/Zn}$, $I_{2/TPW1}$, $I_{2/Sn}$, $I_{2/TPW2}$ | 1.414 x $10^{-3}$ A | 1.6 x $10^{-5}$ A |
| $A_{TPW}$ | -0.73x$10^{-3}$ K $m^{-1}$ | 6 x $10^{-5}$ K $m^{-1}$ |
| $h_{TPW1}$, $h_{TPW2}$ | 187x $10^{-3}$ m | 3 x $10^{-3}$ m |
| $A_{Zn}$ | 2.70x$10^{-3}$ K $m^{-1}$ | 6 x $10^{-5}$ K $m^{-1}$ |
| $h_{Zn}$ | 195 x $10^{-3}$ m | 3 x $10^{-3}$ m |
| $A_{Sn}$ | 2.20x$10^{-3}$ K $m^{-1}$ | 6 x $10^{-5}$ K $m^{-1}$ |
| $h_{Sn}$ | 192 x $10^{-3}$ m | 3 x $10^{-3}$ m |
| $C_{1/TPW1}$, $C_{1/TPW2}$ | 0.000 0 K | 1 x $10^{-4}$ K |
| $C_{1/Zn}$ | 0.000 0 K | 7 x $10^{-4}$ K |
| $C_{1/Sn}$ | 0.000 0 K | 5 x $10^{-4}$ K |
| $C_{2/TPW1}$, $C_{2/TPW2}$ | 0.000 00 K | 0.5 x $10^{-4}$ K |
| $C_{2/Zn}$; $C_{2/Sn}$ | 0.000 0 K | 2 x $10^{-4}$ K |
| $B_{Zn}$ | 4.3 x$10^{-8}$ K $Pa^{-1}$ | 6x$10^{-10}$ K $Pa^{-1}$ |
| $B_{Sn}$ | 3.3 x$10^{-8}$ K $Pa^{-1}$ | 6x$10^{-10}$ K $Pa^{-1}$ |
| $\delta p_{Zn}$, $\delta p_{Sn}$ | 0 Pa | 1x$10^{2}$ Pa |

## 3. Implementation and application of the model

The model presented above was programmed in a computer software environment with symbolic manipulation facilities. Computer processing allowed all variables and their correlations to be included and treated in a consistent manner. The resulting analytical expressions of the sensitivity coefficients are very cumbersome and they cannot be handled other way but by electronic means. For quick reference, the new model and its implementation were labeled CAM (an acronym for Comprehensive Analytical Model).

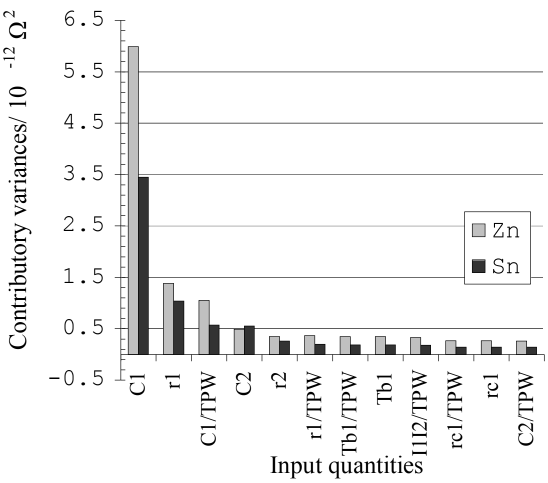


**Figure 1**. Contributory variances to the combined variances $u_c^2(W_{FP})$

The temperature sub-range from 273.15 K to 692.677 K was chosen to illustrate the use of the model, as most of the information available in the reference literature [5], [6], [8], [9], [10] is for this sub-range. The fixed points specified by ITS-90 for the calibration of an SPRT in this sub-range are the TPW (273.16 K) and the freezing points of tin (505.078 K) and of zinc (692.677 K).

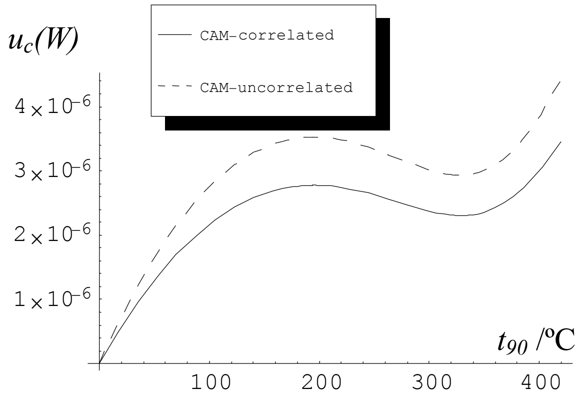


**Figure 2.** Combined standard uncertainties propagated

The set of input data used for this demonstration of CAM is presented in Table 1. Values originate from NIM Bucharest research projects and from the reference literature.

Fig. 1 presents the resulting twelve largest contributory variances of the input quantities to the combined variance $u_c^2(W_{FP})$, evaluated at both Sn and Zn fixed points. Among those contributory variances is the one labeled "$I_1I_2$", associated with the correlated input quantities $I_1$ and $I_2$. The calibration uncertainties $u_c(W_{FP})$ evaluated based on the complete model are 2.706 x $10^{-6}$ at Sn FP, that is 0.73 mK, and 3.449 x $10^{-6}$ at Zn FP, that is 0.99 mK.

It is worth highlighting that contributory variances such as those in Fig. 1 provide a better basis for comparing and monitoring the performances of the national realizations of ITS-90 than the corrections-based parameters currently used in international comparisons (e.g. in EUROMET projects). Some of those corrections have complex expressions that include several independent input quantities, thus obscuring the direct link between the source of the uncertainty and the final effect. The de-composition of the combined variance on the genuine input quantities (defined above) facilitates the identification of those physical parameters that determine the achieved accuracy of the measurement.

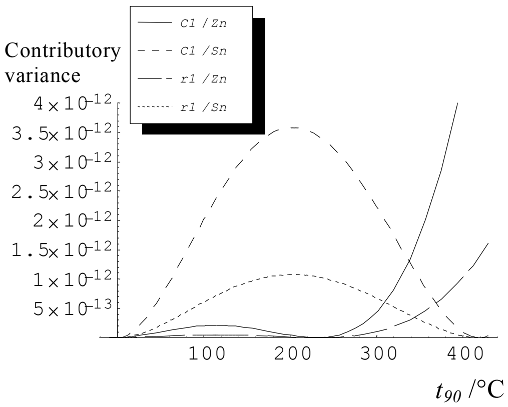


**Figure 3.** Contributory variances over temperature sub-range

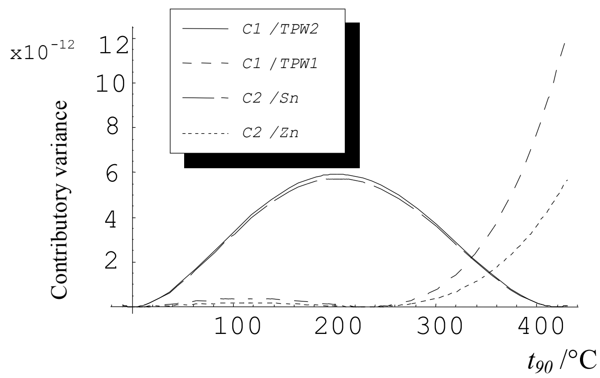


**Figure 4.** Contributory variances over temperature sub-range

The CSU at any intermediate temperature, propagated from the calibration uncertainties at the fixed points, is represented in Fig. 2. The effect of neglecting correlations among input quantities is shown to be approx. 27% of the $u_c(W)$ determined with CAM-correlated, at $T_{Sn}$ and $T_{Zn}$.

For eight input quantities, having the greatest influence over $W$, the variation with temperature of their contributory variances was illustrated in Fig. 3 and 4.

The contributory variances depend directly on the actual values of the standard uncertainties associated with the input estimates. When those uncertainties change, the relative importance of the contributory variances associated to some input quantities, in terms of impact on the combined variance, can also vary a lot or even switch places. CAM enables comprehensive and detailed predictions of such situations, as shown in Fig. 5 for the input estimates $r_{1/TPW1}$ and $T_{b1/TPW1}$. A more in-depth sensitivity analysis of the combined standard uncertainty evaluated with CAM is presented in [7].

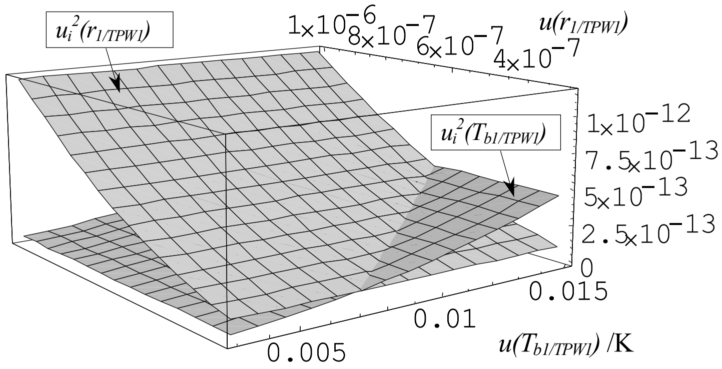


**Figure 5.** Contributory variances $u_i^2$ as functions of the standard uncertainties $u$ associated to input quantities $r_{1/TPW1}$ and $T_{b1/TPW}$ ($T_{90}$ = 623.15 K)

## 4. Conclusions

The paper briefly presents the main features and advantages of a new model for evaluating the combined standard uncertainty at any intermediate temperature, from the SPRT calibration uncertainties at the defining fixed points of ITS-90.

By replacing the classical modularization based on corrections with an unique expression of the model function, that integrates all "genuine" (i.e. elementary) input quantities, two important gains were demonstrated:

- the impact of correlations was minimized and a direct and consistent evaluation of the combined standard uncertainty was enabled, with all correlations allowed for;
- the contributory variances of each elementary input quantity were shown out, so that a direct link between the accuracy of measurements and the manageable physical factors involved can be drawn.

These advantages make the proposed model a powerful tool, with immediate application in international comparisons of ITS-90 realizations, as well as in other, lower level, temperature measurement activities that require advanced uncertainty monitoring.

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